CHAPTER TWO

Knots and Borromean Rings

THREE CURIOUSLY INTERLOCKED RINGS, familiar to many people in this country as the trade-mark of a popular brand of beer, are shown in Figure 5. Because they appear in the coat of arms of the famous Italian Renaissance family of Borromeo they are sometimes called Borromean rings. Although the three rings cannot be separated, no two of them are linked. It is easy to see that if any one ring is taken from the set, the remaining two are not linked.

In a chapter on paper models of topological surfaces, which appears in the first *Scientific American Book of Mathematical Puzzles & Diversions,* I mentioned that I knew of no paper model of a single surface, free of self-intersection, that has three edges linked in the manner of the Borromean rings. "Perhaps," I wrote, "a clever reader can succeed in constructing one."

This challenge was first met in the fall of 1959 by David A. Huffman, associate professor of electrical engineering at the Massachusetts Institute of Technology. Huffman not only succeeded in making models of several different types of surface with Borromean edges; in doing so he also hit upon some beautifully simple methods by which one can construct

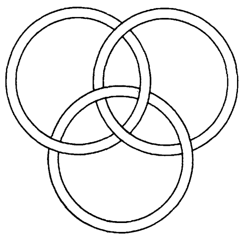


Figure 5
The three Borromean rings

paper models of a surface with edges that correspond to any type of knot or set of knots—interlaced, interwoven or linked in any manner whatever. Later he discovered that essentially the same methods have been known to topologists since the early 1930's, but because they had been described only in German publications they had escaped the attention of everyone except the specialists.

Before applying one of these methods to the Borromean rings, let us see how the method works with a less complex structure. The simplest closed curve in space is, of course, a curve that is not knotted. Mathematicians sometimes call it a knot with zero crossings, just as they sometimes call a straight line a curve with zero curvature. Diagram 1 in Figure 6 is such a curve. The shaded area in the diagram represents a two-sided surface whose edge corresponds to the curve. It is easy to cut the surface out of a sheet of paper. The actual shape of the cutout does not matter, because we are interested only in the fact that its edge is a simple closed curve. But there is another way to color the diagram. We can color the *outside* of the curve *(diagram 2 in Figure 6)* and imagine that the diagram is on the surface of a sphere. Here the closed curve surrounds a *hole* in the sphere. The two models—the first cutout and the sphere with the hole—are topologically equivalent. When put together edge to edge, they form the closed, two-sided surface of a sphere.

Now let us try the same method on a slightly more complicated diagram (*diagram 3*) of the same space curve. Think of this curve as a piece of rope. At the crossing we indicate that

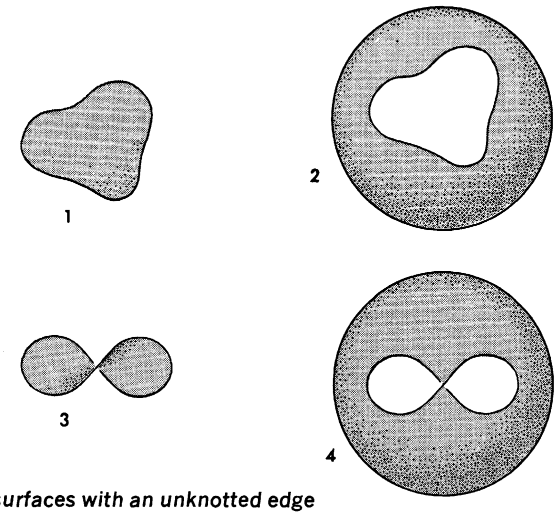


Figure 6
Models of surfaces with an unknotted edge

one segment of rope passes under the other, like a highway underpass, by breaking the line as shown. This curve also is a knot of zero crossings, because it can be manipulated so that the crossing is eliminated. (The order of a knot is the minimum number of crossings to which the knot can be reduced by deformation.) As before, we shade the diagram with two colors, tinting it so that no two regions with a common boundary have the same color. This can always be done in two different ways, one a reverse print of the other.

If we color diagram 3 as shown in the illustration, the model is merely a sheet of paper with a half twist. It is two-sided and topologically equivalent to each of the previous models. But when we color the diagram in the alternate way *(diagram 4)*, regarding the white spaces as holes in a sphere, we obtain a surface that is a Möbius strip. It too has an edge that is a knot of zero crossings (that is, not a knot), but now the surface is one-sided and topologically distinct from the preceding model. The closed, no-edged surface that results when the two models are fitted together is a cross cap, or projective plane: a one-sided surface that cannot be constructed without self-intersection.

The same general procedure can be applied to the diagram of any knot or group of knots, linked together in any manner. Let us see how it applies to the Borromean rings. The first step is to map the rings as a system of underpasses, making sure that no more than two roads cross at each pass. Next, we color the map in the two ways possible *(diagrams 1 and 2* *in Figure 7)*. Each crossing represents a spot where the paper surface (the shaded areas) is given a half twist in the direction indicated. The one-sided surface shown in diagram 1 is easily made with paper, either in the elegant symmetrical form shown or in topologically equivalent forms such as the one depicted in diagram 3. The model that results from diagram 2, with the Borromean rings outlining the holes in a sphere, seems at first glance quite different from the preceding model. Actually it is topologically the same. Sometimes the two methods of coloring lead to equivalent models, sometimes not.

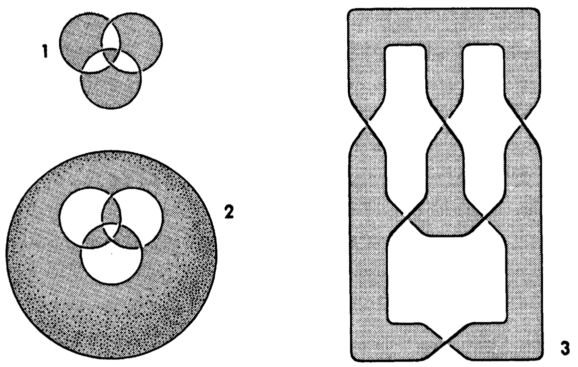


Figure 7
Topologically equivalent one-sided surfaces with Borromean-ring edges

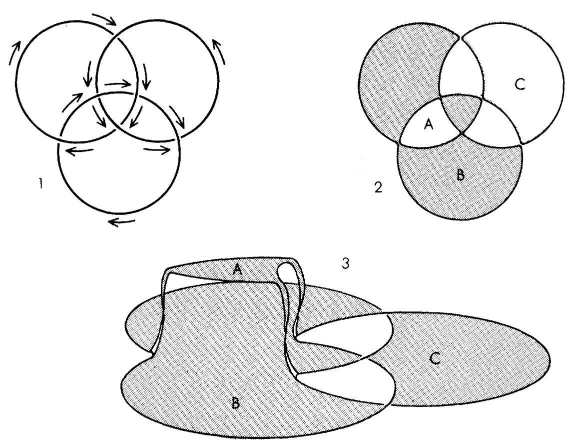
It can be proved that this double procedure can be applied to any desired knot or group of knots, of any order, linked together in any manner. Most models obtained in this way, however, turn out to be one-sided. Sometimes it is possible to rearrange the crossings of the diagram so as to yield a two-sided surface, but usually it is extremely difficult to see how to make this sort of modification. The following method, also rediscovered by Huffman, guarantees a two-sided model.

To illustrate the procedure, let us apply it to the Borromean rings. First draw the diagram, but with light pencil lines. Place the point of the pencil on any one of the curves and trace it around, in either direction, back to the starting point. At each crossing make a small arrow to indicate the direction in which you are traveling. Do the same with each of the other two curves. The result is diagram 1 in Figure 8.

Now go over this diagram with a heavier pencil or crayon, starting at any point and moving in the direction of the arrows for that curve. Each time you come to a crossing turn either right or left as indicated by the arrows on the intersecting strand. Continue along the other strand until you reach another crossing, then turn again, and so on. It is as if you were driving on a highway and each time you reached an underpass or overpass you leaped to the other road and continued in the direction its traffic was moving. You are sure to return to your starting point after tracing out a simple closed curve. Now place the crayon at any other point on the diagram and repeat the procedure. Continue until you have gone over the entire diagram. Interestingly enough, the closed paths produced in this way will never intersect one another. In this case the result will look like diagram 2 in Figure 8.

Each closed curve represents an area of paper. Where two areas are alongside each other, the touching points represent half twists (in the direction indicated on the original diagram) that join the areas. Where one area is *inside* another, the smaller area is regarded as being above the larger, like two floor levels in a parking garage. The touching points represent half twists, but now the twists must be thought of as twisted ramps that join the two levels. The finished model is shown at 3 in Figure 8; it is two-sided and its three edges are Borromean. It can be proved that any model constructed by this procedure will be two-sided. This means that it can

Figure 8
Steps in making a two-sided surface with Borromean-ring edges



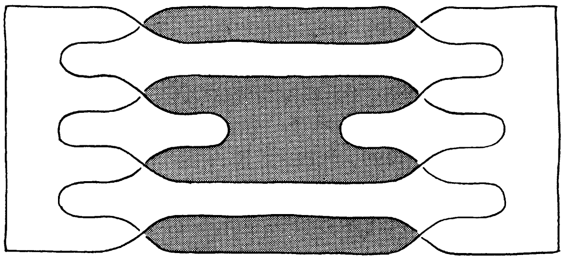


Figure 9
A two-sided, Borromean-ring-edged surface

be painted in two contrasting colors, or constructed from paper that is differently colored on its two sides, without having one color run into the other. Figure 9, supplied by Huffman, shows a pleasingly symmetrical way of diagraming such a surface.

The reader may enjoy building models of other knots and linkages. The figure-of-eight knot, for example, leads to very pleasing, symmetrical surfaces. The first diagram in Figure 10 is one way in which this familiar knot can be mapped. Diagrams of this sort, by the way, are used in knot theory for determining the algebraic expression for a given knot. Equivalent knots, in the sense that one can be deformed into the other, have the same algebraic formula, but not all knots with the same formula are equivalent. It is always assumed, of course, that the knots are tied in closed curves in three-dimensional space. Knots in ropes open at the ends, or in closed curves in four-dimensional space, can all be untied and are therefore equivalent to no knots at all.

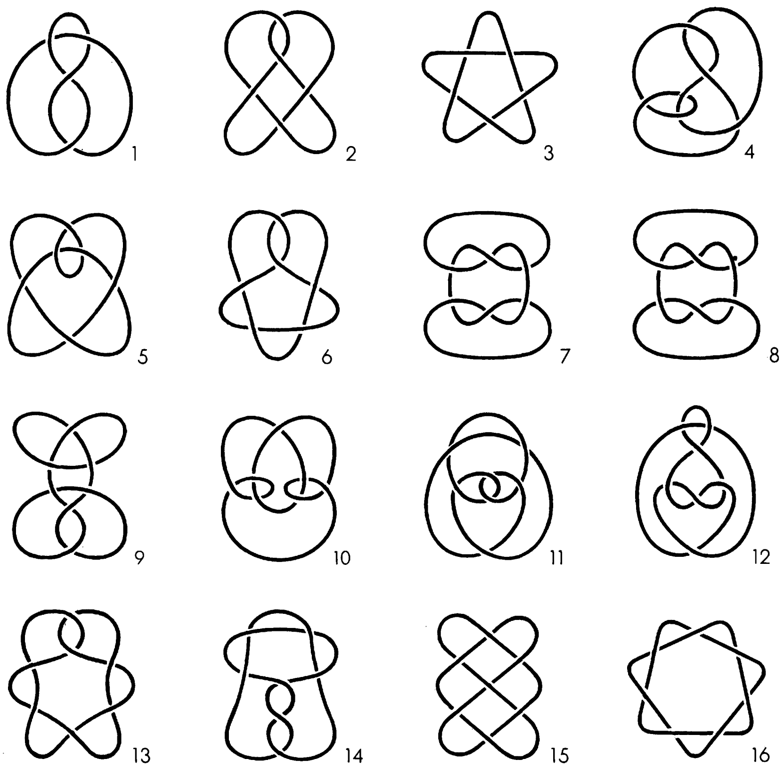
The figure-of-eight knot is the only knot that reduces to a minimum of four crossings, just as the overhand or trefoil knot is the only type that has a minimum of three crossings. Unlike the trefoil, however, the figure-of-eight knot has no mirror image, or rather it can be deformed into its mirror image. Such knots are called "amphicheiral," meaning that they "fit either hand," like a rubber glove that can be turned inside out.

No knots are possible with one or two crossings. There are two five-crossers, five six-crossers, eight seven-crossers *(see*

Figure 10). This tabulation does not include mirror-image knots but does include knots that can be deformed into two simpler knots side by side. Thus the square knot *(knot 7 in the illustration)* is the "product" of a trefoil and its mirror image; the granny *(knot 8)* is the "product" of two trefoils of the same handedness. Knots 3 and 16 have very simple models. You have only to give a strip five half twists and join the ends to make its edge form knot 3, seven half twists to make it form knot 16.

All sixteen of these knots can be diagramed so that their crossings are alternately over and under. (Only knot 7, the square knot, is shown in nonalternating form.) Not until the number of crossings reach eight is it possible to construct knots (there are three) that cannot be diagramed in alternating form.

Figure 10
Knots of four crossings (1), five crossings (2, 3), six crossings (4–8) and seven (9–16)



The reader may wonder why knot 9, a combination of a trefoil and a figure-of-eight, does not have two distinct forms like the square knot and granny, knots 7 and 8, each of which combines two trefoils. The answer is that the figure-of-eight part of knot 9 can be transformed to its mirror image without altering the handedness of the trefoil part, therefore there is only the knot shown and its mirror image.

A knot that cannot be deformed into simpler knots side by side is called a prime knot. All the knots in the illustration are prime except 7, 8 and 9. Knots have been carefully tabulated up through ten crossings, but no formula has yet emerged by which the number of different knots, given n crossings, can be determined. The number of prime knots with ten crossings is thought to be 167. Only wild guesses can be made as to the number of prime knots with eleven and twelve crossings.

Like topology, to which it obviously is closely related, the theory of knots is riddled with unsolved, knotty problems. There is no general method known for deciding whether or not any two given knots are equivalent, or whether they are interlocked, or even for telling whether a tangled space curve is knotted or not. To illustrate the latter difficulty, I have concocted the puzzle depicted in Figure 11. This strange-looking surface is one-sided and one-edged, like a Möbius strip, but is the edge knotted? If so, what kind of knot is it? The reader is invited to study the picture, make a guess, then test his guess by the following empirical method. Construct the surface with paper and cut it along the broken line. This will produce one single strip that will be tied in the same type of knot as the edge of the original surface. By manipulating the strip carefully so as not to tear the paper you can reduce it to its simplest form and see if your guess is verified. The result may surprise you.

In the 1860's the British physicist William Thomson (later Lord Kelvin) developed a theory in which atoms are vortex rings in an incompressible, frictionless, all-pervading ether. J. J. Thomson, another British physicist, later suggested that molecules might be the result of various knots and linkages of Lord Kelvin's vortex rings. This led to a flurry of interest in topology on the part of physicists (notably the Scottish

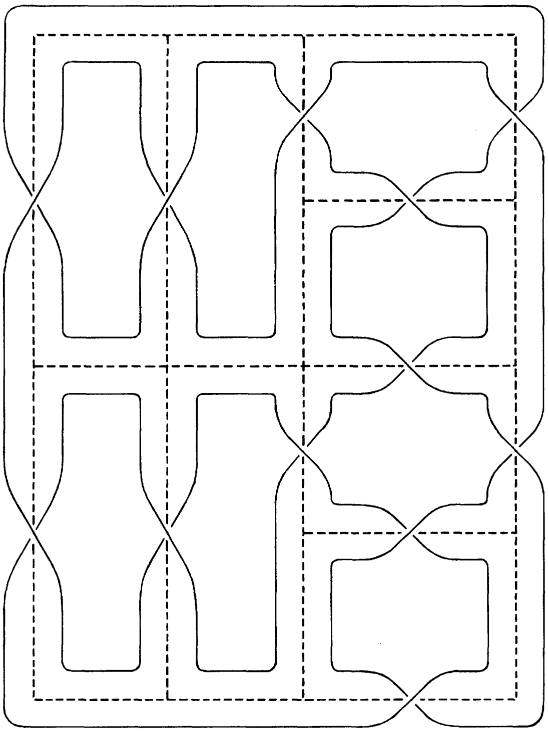


Figure 11
A one-sided, one-edged surface. Is the edge knotted?

physicist Peter Guthrie Tait), but when the vortex theory was discarded, the interest waned. Perhaps it will revive now that chemists at the Bell Telephone Laboratories have produced radically new compounds, called catenanes, that consist of carbon molecules in the form of rings that are actually linked. It is now theoretically possible to synthesize compounds made up of closed chains that can be knotted and interlocked in bizarre ways. (See Edel Wasserman, "Chemical Topology," *Scientific American,* November 1962, pages 94–102.) Who can guess what outlandish properties a carbon compound might have if all its molecules were, say, figure-of-eight knots? Or if its molecules were joined into triplets, each triplet interlocked like a set of Borromean rings?

One might suppose that living organisms would be free of knots, but such is not the case. Thomas D. Brock, a microbiologist at Indiana University, reported in *Science,* Vol. 144, No. 1620 (May 15, 1964), pages 870–72, on his discovery of a stringlike microbe that reproduces by tying itself in a knot (the knot can be an overhand, figure-of-eight, granny, or some other simple knot) which pulls tighter and tighter until the knot fuses into a bulb and free ends of the filament break off to form new microbes. And if the reader will check David Jensen's fascinating article on "The Hagfish" *(Scientific American,* February 1966, pages 82–90), he will learn about an eel-like fish that cleans itself of slime and does other curious things by tying itself into an overhand knot.

What about humans? Do they ever tie parts of their anatomy into knots? The reader is invited to fold his arms and give the matter some thought.

ANSWERS

If the surface shown in Figure 11 is constructed with paper and cut as explained, the resulting endless strip will be free of any knot. This proves that the surface's single edge is similarly unknotted. The surface was designed so that its edge corresponds to a pseudo knot known to conjurers as the Chefalo knot. It is formed by first tying a square knot, then looping one end twice through the knot in such a way that when both ends are pulled, the knot vanishes.